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Decomposition of vector fields by scalar potentials

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Abstract. Details of a decomposition of an arbitrary vector field by scalar potentials are given. Known expansions for divergence-free fields, curl-free fields and transverse fields are shown to be special cases.

In some physical problems involving vector fields in three dimensions a particular point, $\mathbf{0}$, the origin, is distinguished on physical or mathematical grounds. This is the situation in electromagnetism when multipole expansions are made about some convenient point in or near the sources of the fields. In these cases a useful decomposition of an arbitrary vector field $\boldsymbol{A}(\boldsymbol{x})$ is (proof below)

$$\boldsymbol{A}(\boldsymbol{x})=\boldsymbol{x}\times\nabla P(\boldsymbol{x})+\nabla Q(\boldsymbol{x})+\boldsymbol{x}R(\boldsymbol{x}). \tag{1}$$

There is a small amount of arbitrariness in the choice of the scalar potentials P, Q and R. To $P(\boldsymbol{x})$ may be added an arbitrary function of $r\equiv|\boldsymbol{x}|$, and a similar change in Q could be compensated by a change in R:

$$\nabla Q+\boldsymbol{x}R=\nabla(Q+f(r))+\boldsymbol{x}(R-r^{-1}f').$$

But if we impose the extra condition on P and Q that their average values on spheres with centre $\mathbf{0}$ vanish,

$$\int \mathrm{d}\Omega P(r,\Omega)=\int \mathrm{d}\Omega Q(r,\Omega)=0, \tag{2}$$

then P, Q and R are unique. They are determined on each sphere $r=|\boldsymbol{x}|$ by invariant differential functions of the field $\boldsymbol{A}$ on the same sphere. They vanish on any sphere where the derivatives of $\boldsymbol{A}$ vanish.

If the divergence of $\boldsymbol{A}$ is zero, $\nabla\cdot\boldsymbol{A}=0$, then (1) can be converted (see below) to the form

$$\boldsymbol{A}(\boldsymbol{x})=\boldsymbol{x}\times\nabla P(\boldsymbol{x})+\nabla\times(\boldsymbol{x}\times\nabla)S(\boldsymbol{x}) \tag{3}$$

in which the potentials P and S are unique if they are required to satisfy

$$\int \mathrm{d}\Omega P=\int \mathrm{d}\Omega S=0. \tag{4}$$

Equation (3) provides a uniquely definable vector potential for a divergence-free field:

$$\boldsymbol{A}(\boldsymbol{x})=\nabla\times(-\boldsymbol{x}P(\boldsymbol{x})+\boldsymbol{x}\times\nabla S(\boldsymbol{x})). \tag{5}$$

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An advantage of the vector potential in (5) over the one commonly used in electromagnetism,

$$A(x) = \nabla \times \left(\frac{1}{4\pi} \int \mathrm{d}x' \frac{\nabla' \times A(x')}{|x - x'|} \right), \tag{6}$$

is that the nonlocality in the former is more limited than in the latter. The potentials $P(x)$ and $S(x)$ are determined by A on the sphere $r \equiv |x|$, whereas the integral in (6) gets contributions from all points where $\nabla \times A \neq 0$. The divergence of the vector potential in (6) is zero whereas this is not generally true of the potential in (5).

If the curl of A is zero, then

$$\nabla \times A = \nabla \times (x \times \nabla) P - x \times \nabla R = 0.$$

This equation has the form (3), and the uniqueness of the potentials subject to (2) and (4) here implies that

$$P(x) = 0, \qquad R(x) = R(r).$$

Therefore, from (1),

$$A(x) = \nabla \left(Q(x) + \int_0^r r' R(r')\, \mathrm{d}r' \right)$$

in which the scalar potential is split up into a part (Q) determined by A on the sphere $r = |x|$ and averaging to zero over the surface, and a part getting contributions from A inside the sphere.

The decomposition (3), for a solenoidal field, has a long history. The potentials P and S are essentially Debye potentials, or radial Hertz potentials. Relatively recent references, from which others may be traced back, are Bouwkamp and Casimir (1954), Nisbet (1955) and Wilcox (1957).

A decomposition of an arbitrary transverse vector field has been given by Wilcox (1957), Lomont and Moses (1961), Keller (1961) and Moses (1976). A transverse field, one for which $x \cdot A = 0$, may be decomposed

$$A = A_1 + A_2 \tag{7a}$$

where

$$(x \times \nabla) \times A_1 = -A_1, \qquad (x \times \nabla) \cdot A_2 = 0. \tag{7b}$$

Equation (1) may be rewritten

$$A = x \times \nabla P - \frac{x \times (x \times \nabla) Q}{r^2} + \hat{x}\hat{x} \cdot \nabla Q + xR. \tag{8a}$$

and if it is then specialized to transverse A,

$$A_1 \equiv x \times \nabla P, \qquad A_2 \equiv -\frac{x \times (x \times \nabla Q)}{r^2} \tag{8b}$$

have the required properties. In fact, for an arbitrary A, the same definition of A_1 and $A_2 \equiv \nabla Q + xR$, the relations (7b) are true.

Gray (1978) (see also Elsasser 1946) has given an expansion of an arbitrary field

$$\boldsymbol{A} = \boldsymbol{x}\times\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{\nabla})\chi + \boldsymbol{\nabla}\phi$$
$$= \boldsymbol{x}\times\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\left(\phi - \frac{\partial}{\partial r}(r\chi)\right) + \boldsymbol{x}\nabla^2\chi.$$

Comparison with (1) shows that some arbitrariness in ϕ and χ remains.

One of the main advantages in obtaining a decomposition of the form (1) for a field $\boldsymbol{A}$ is that the potentials are coordinate independent; but if we use a spherical polar coordinate system with centre $\mathbf{0}$ to try to solve (1) we get

$$A_r\boldsymbol{e}_r + A_\theta\boldsymbol{e}_\theta + A_\phi\boldsymbol{e}_\phi$$
$$= \boldsymbol{e}_r\left(\frac{\partial Q}{\partial r} + rR\right) + \boldsymbol{e}_\theta\left(-\frac{1}{\sin\theta}\frac{\partial P}{\partial\phi} + \frac{1}{r}\frac{\partial Q}{\partial\theta}\right) + \boldsymbol{e}_\phi\left(\frac{\partial P}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial Q}{\partial\phi}\right).$$

This gives

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Q}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Q}{\partial\phi^2} = \frac{r}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{\partial A_\phi}{\partial\phi}\right] \tag{9}$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 P}{\partial\phi^2} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right], \tag{10}$$

whose coordinate-independent forms, which may be obtained directly from (1), are

$$(\boldsymbol{x}\times\boldsymbol{\nabla})^2 Q = \boldsymbol{x}\,.\,\boldsymbol{\nabla}\times(\boldsymbol{x}\times\boldsymbol{A}) = \boldsymbol{x}\times\boldsymbol{\nabla}\,.\,\boldsymbol{x}\times\boldsymbol{A} \tag{11}$$

and

$$(\boldsymbol{x}\times\boldsymbol{\nabla})^2 P = \boldsymbol{x}\,.\,\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{x}\times\boldsymbol{\nabla}\,.\,\boldsymbol{A} \tag{12}$$

The operator $(\boldsymbol{x}\times\boldsymbol{\nabla})^2 = r^2\nabla^2 - (\partial/\partial r)r^2(\partial/\partial r)$ is written $-L^2$ in quantum mechanics.

The solutions of (11) and (12) are unique up to additive functions of r; those satisfying the extra condition (2) are (Courant and Hilbert 1953, Wilcox 1957)

$$Q(r,\hat{\boldsymbol{x}}) = \frac{1}{4\pi}\int \mathrm{d}\Omega'\,\boldsymbol{x}'\,.\,\boldsymbol{\nabla}'\times(\boldsymbol{x}'\times\boldsymbol{A}(r',\hat{\boldsymbol{x}}'))\,\lg(1-\hat{\boldsymbol{x}}\,.\,\hat{\boldsymbol{x}}') \tag{13}$$

where $r' = r$, and

$$P(r,\hat{\boldsymbol{x}}) = \frac{1}{4\pi}\int \mathrm{d}\Omega'\,\boldsymbol{x}'\,.\,\boldsymbol{\nabla}'\times\boldsymbol{A}\,\lg(1-\hat{\boldsymbol{x}}\,.\,\hat{\boldsymbol{x}}'). \tag{14}$$

To verify (2) one must use the integrability conditions

$$\int \mathrm{d}\Omega'\,\boldsymbol{x}'\,.\,\boldsymbol{\nabla}'\times\boldsymbol{A}(\boldsymbol{x}'\times\boldsymbol{A}) = \int \mathrm{d}\Omega'\,\boldsymbol{x}'\,.\,\boldsymbol{\nabla}'\times\boldsymbol{A} = 0, \tag{15}$$

which follows from the right-hand sides of (9) and (10), and the periodicity in ϕ of A_θ and A_ϕ, or Stokes' theorem with the right-hand sides of (11) and (12).

One notices that the values of Q and P on a sphere r depend only on the values of simple differential invariants of $\boldsymbol{A}$ on the same sphere. Given a point $\boldsymbol{x}$, if one uses spherical polar coordinates whose pole is $\hat{\boldsymbol{x}}$ in the integral (13), one has

$$Q(r, \hat{\boldsymbol{x}}) = \frac{r}{4\pi} \int \mathrm{d}\theta' \, \mathrm{d}\phi' \left(\frac{\partial}{\partial \theta'} (\sin \theta' \, A_{\theta'}) + \frac{\partial A_{\phi'}}{\partial \phi'} \right) \lg(1 - \cos \theta')$$

$$= -\frac{r}{4\pi} \int \mathrm{d}\theta' \, \mathrm{d}\phi' \, (1 + \cos \theta') A_{\theta'}. \tag{16}$$

The similar expression for P, using the same coordinates in the integral, is

$$P(r, \hat{\boldsymbol{x}}) = -\frac{1}{4\pi} \int \mathrm{d}\theta' \, \mathrm{d}\phi' \, (1 + \cos \theta') A_{\phi'}. \tag{17}$$

Equations (16) and (17) express Q and P as averages of the longitudinal and latitudinal components, with respect to the pole $\hat{\boldsymbol{x}}$, of the field $\boldsymbol{A}$.

In order to prove that the form (1) is possible, one introduces a new field

$$\boldsymbol{B}(\boldsymbol{x}) \equiv \boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{x} \times \nabla P - \nabla Q$$

where P and Q are defined by (14) and (13). The field $\boldsymbol{B}$ satisfies

$$\boldsymbol{x} \cdot \nabla \times B = \boldsymbol{x} \cdot \nabla \times (\boldsymbol{x} \times \boldsymbol{B}) = 0,$$

and from these equations one can show, in a familiar manner (Bouwkamp and Casimir 1954, Backus 1958) that $B_\theta = B_\phi = 0$. Hence $\boldsymbol{B}$ is purely radial.

A direct proof of (1) shows how the formulae work. We must verify that the transverse components of $\boldsymbol{A}$ are given by the transverse components of $\boldsymbol{x} \times \nabla P + \nabla Q$. Without loss of generality, we can choose Cartesian coordinates such that the point of comparison is $(0, 0, z = r)$ and the direction of the transverse component is parallel to the x axis. We then have to compare A_x with $\partial Q/\partial x - r\partial P/\partial y$, or, using spherical polars, with

$$\frac{1}{r} \frac{\partial Q}{\partial \theta}(\theta = 0, \phi = 0) - \frac{\partial P}{\partial \theta}(\theta = 0, \phi = \tfrac{1}{2}\pi). \tag{18}$$

Using $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}' = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$, we can differentiate once under the integral sign in (13) and (14), and the expression (18) becomes

$$\frac{1}{4\pi} \int \mathrm{d}\theta' \, \mathrm{d}\phi' \left(\frac{\partial}{\partial \theta'} (\sin \theta' \, A_{\theta'}) + \frac{\partial A_{\phi'}}{\partial \phi'} \right) \frac{(-\sin \theta' \cos \phi')}{1 - \cos \theta'}$$

$$-\frac{1}{4\pi} \int \mathrm{d}\theta' \, \mathrm{d}\phi' \left(\frac{\partial}{\partial \theta'} (\sin \theta' \, A_{\phi'}) - \frac{\partial A_{\theta'}}{\partial \phi'} \right) \frac{(-\sin \theta' \sin \phi')}{1 - \cos \theta'}. \tag{19}$$

The integrals in (19) are the limits, as $\epsilon \to 0$, of integrals with the region $\theta' < \epsilon$ excluded. After integrating by parts and taking the benefit of cancellations one gets

$$\lim_{\epsilon \to 0} \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \, (1 + \cos \epsilon)(A_{\theta'} \cos \phi' - A_{\phi'} \sin \phi')$$

$$= \lim_{\epsilon \to 0} \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \, (1 + \cos \epsilon) A_x(\epsilon r \cos \phi', \epsilon r \sin \phi', r)$$

$$= A_x(0, 0, r).$$

Once the form (1) is established, we can calculate the potential $R(\boldsymbol{x})$. Integrating (1) over the surface of a sphere, and using (2) and the divergence theorem, we get

$$\int \mathbf{d}\boldsymbol{S} \cdot \boldsymbol{A} = r^3 \int \mathrm{d}\Omega\, R = \int \mathrm{d}V\, \boldsymbol{\nabla} \cdot \boldsymbol{A}. \tag{20}$$

Taking the limit as the radius of the sphere tends to zero,

$$R(\mathbf{0}) = \tfrac{1}{3}\boldsymbol{\nabla} \cdot \boldsymbol{A}(\mathbf{0}). \tag{21}$$

Operating on (1) with $\boldsymbol{x} \times \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times$ gives

$$(\boldsymbol{x} \times \boldsymbol{\nabla})^2 R = -\boldsymbol{x} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}), \tag{22}$$

whose solution (we bear (20) in mind) is

$$R(\boldsymbol{x}) = \frac{1}{4\pi r^2} \int \mathrm{d}\Omega'\, \boldsymbol{x}' \cdot \boldsymbol{A}(r, \hat{\boldsymbol{x}}') - \frac{1}{4\pi} \int \mathrm{d}\Omega'\, \boldsymbol{x}' \cdot \boldsymbol{\nabla}' \times (\boldsymbol{\nabla}' \times \boldsymbol{A})\, \lg(1 - \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}'). \tag{23}$$

As (21) shows, the apparent singularity at $r = 0$ is not present.

If $\boldsymbol{A}$ has zero divergence, (1) may be reduced to the simpler form (3). The condition $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ implies

$$\nabla^2 Q + \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 R) = 0. \tag{24}$$

This relation allows

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{x} \times \boldsymbol{\nabla}) S &= \boldsymbol{x} \nabla^2 S - \boldsymbol{\nabla} \frac{\partial}{\partial r}(rS) \\ &= \boldsymbol{\nabla} Q + \boldsymbol{x} R \end{aligned}$$

to be solved. We define

$$S = -\frac{1}{r} \int_0^r Q\, \mathrm{d}r, \tag{25}$$

which satisfies (4), and then

$$\begin{aligned} \nabla^2 S &= \left(\frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{1}{r^2} (\boldsymbol{x} \times \boldsymbol{\nabla})^2 \right) S \\ &= -\frac{1}{r^3} \int_0^r r^2 \nabla^2 Q\, \mathrm{d}r = R \end{aligned}$$

in view of (24).

The potential S appears, from (25), to depend on $\boldsymbol{A}$ inside the sphere $r = |\boldsymbol{x}|$. But since $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, the right-hand side of (11) may be expressed as $-(\partial/\partial r)(r^2 A_r)$, and Q itself may be written as the derivative of a function of r. Thus S depends only on the values of $\boldsymbol{x} \cdot \boldsymbol{A}$ for $r = |\boldsymbol{x}|$. This fact is also clear from (3). Since S satisfies

$$(\boldsymbol{x} \times \boldsymbol{\nabla})^2 S = \boldsymbol{x} \cdot \boldsymbol{A}, \tag{26}$$

we have

$$S(\boldsymbol{x}) = \frac{1}{4\pi} \int \mathrm{d}\Omega'\, \boldsymbol{x}' \cdot \boldsymbol{A}\, \lg(1 - \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}'). \tag{27}$$

One can expand $\lg(1-\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}')$ in Legendre polynomials,

$$\lg(1-\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}')=(\lg 2-1)-\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}'),$$

and use the addition theorem for spherical harmonics to convert (27) to

$$S(\boldsymbol{x})=-\sum_{lm}{}'\frac{Y_{lm}(\hat{\boldsymbol{x}})}{l(l+1)}\int \mathrm{d}\Omega'\,\boldsymbol{x}'\cdot\boldsymbol{A}(r,\hat{\boldsymbol{x}}')Y^*_{lm}(\hat{\boldsymbol{x}}'). \tag{28}$$

The prime on the sum indicates that the $l=0$ term is absent; this is because

$$\int \mathrm{d}\Omega'\,\boldsymbol{x}'\cdot\boldsymbol{A}=0 \qquad \text{when } \boldsymbol{\nabla}\cdot\boldsymbol{A}=0.$$

Similar expansions in spherical harmonics can be obtained for P, Q and R from (14), (13) and (23). In fact, these expansions arise as the formal solutions of (12), (11) and (22) using

$$(\boldsymbol{x}\times\boldsymbol{\nabla})^2 Y_{lm}(\hat{\boldsymbol{x}})=-l(l+1)Y_{lm}(\hat{\boldsymbol{x}}).$$

When the vector fields are written in terms of potentials expanded in spherical harmonics, we have expansions in vector spherical harmonics. Though such expansions can be used to establish the decompositions of vector fields, they are not necessary.

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